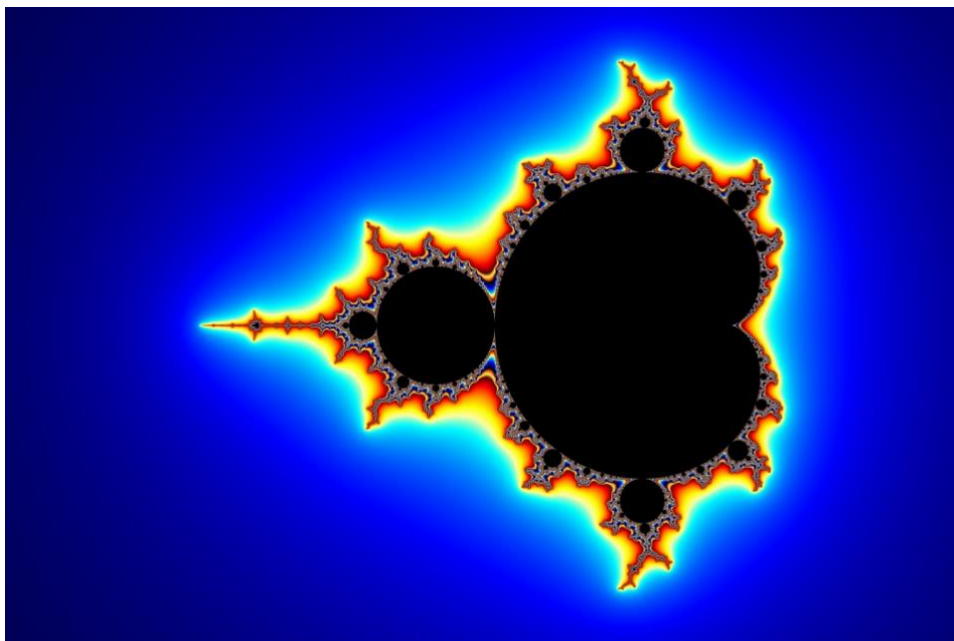


# An Introduction to Fractals

with



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<https://www.mmu.ac.uk/computing-and-maths/staff/profile/dr-stephen-lynch>

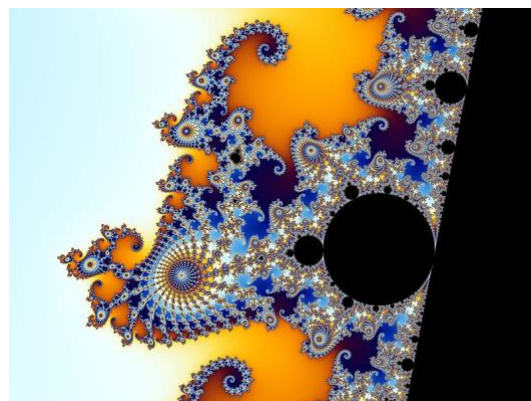
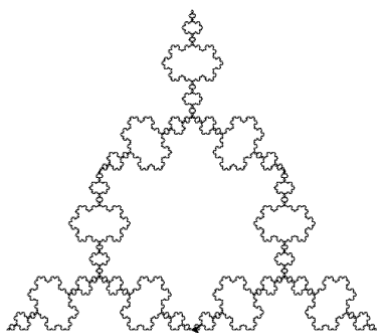
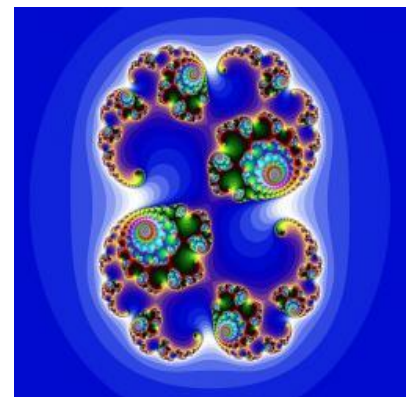
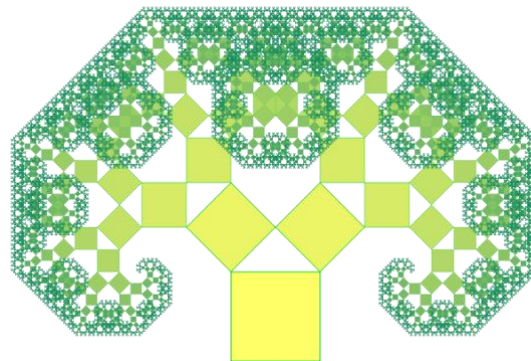
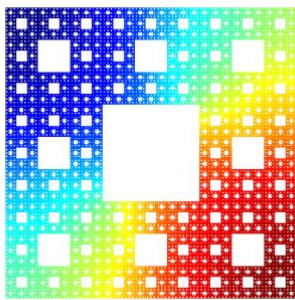
**Definition 1.** A *fractal* is an image repeated on an ever-reduced scale.

**Definition 2.** A *fractal* is an object with non-integer dimension.

### Fractals in Nature




### Mathematical Fractals



**Example 1.** The *Cantor set* (1870).

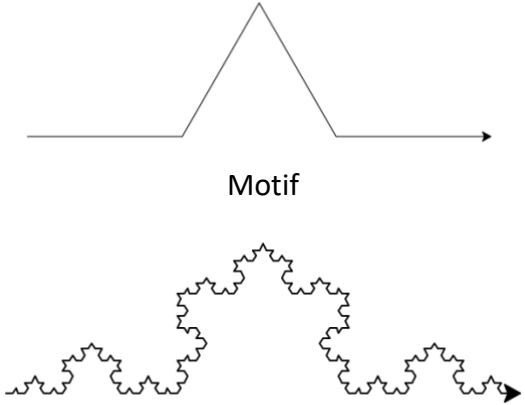
Start with a unit line segment and at each stage remove the middle third segment and replace one segment with two segments each of length one third of the previous segment.

|  |   |
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| <pre> IDLE File Edit Format Run Options Window Help cantor.py - /Users/99900361/Documents/2022 Files/Stephen/MMU... # Program 1: Cantor Fractal. # Run Module and type &gt;&gt;&gt; cantor(-400, 200 , 400) in Python Shell. from turtle import *      # Load functions from turtle module. speed(0)                  # Fastest speed of turtle. def cantor(x, y, length):     if length &gt;= 5:      # Stop when length &lt; 5.         penup()         pensize(3)         pencolor('blue')         setpos(x , y)         pendown()         fd(length)         y -= 60          # Height difference.         cantor(x , y, length / 3)         cantor(x + 2 * length / 3, y, length / 3)         penup()         setpos(x , y + 60) </pre> |  <p><b>Figure 1:</b> The Cantor set to stage 4.</p> <p>To understand the program, set speed(1) and plot the Cantor set to stage 2. Mimic by-hand what the program is doing.</p> |
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**Problem 1:** Edit Program 1 to plot a variation of the Cantor set, where two segments (each one-fifth the length of the previous segment) are removed at each stage.

**Example 2.** The *Koch curve* (1904).

Start with a unit line segment, remove the middle third segment and replace one segment with four segments each of length one third the previous segment, as illustrated below.

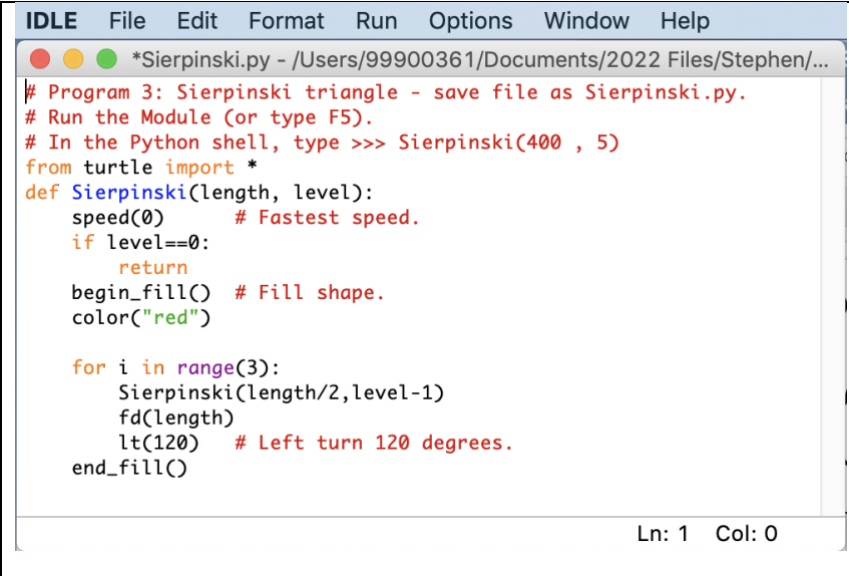
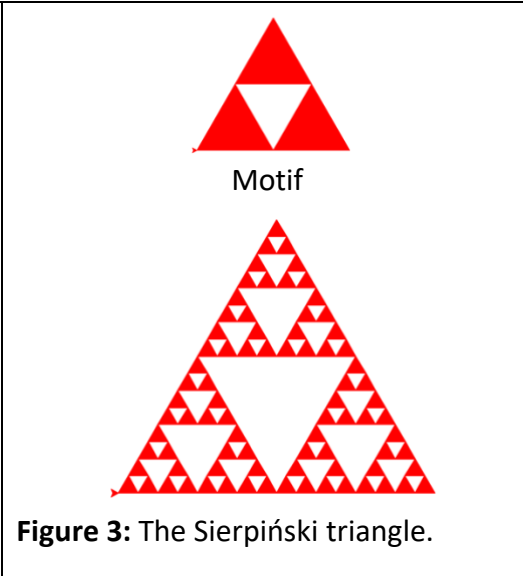
|   |  |
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| <pre> IDLE File Edit Format Run Options Window Help *KochCurve.py - /Users/99900361/Documents/2022 Files/Stephe... # Program 2: Koch Curve Fractal # Run Module and type &gt;&gt;&gt; Koch_Curve(300,4) in Python Shell. from turtle import * speed(0)                  # Fastest speed. def KochCurve(length,level):     if level==0:         fd(length)         return     KochCurve(length/3,level-1)     lt(60)                # Left turn 60 degrees.     KochCurve(length/3,level-1)     rt(120)              # Right turn 120 degrees.     KochCurve(length/3,level-1)     lt(60)     KochCurve(length/3,level-1) </pre> |  <p><b>Figure 2:</b> The Koch curve.</p> |
|---|--|

**Problem 2:** Edit Program 2 to plot a Koch square curve, where one segment is replaced with five segments each one-third the length of the segment before.



**Example 3.** The *Sierpiński triangle* (1915).

Start with a solid equilateral triangle and at each stage remove the middle-inverted triangle. Complete the construction up to stage 4.

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|  <pre># Program 3: Sierpinski triangle - save file as Sierpinski.py. # Run the Module (or type F5). # In the Python shell, type &gt;&gt;&gt; Sierpinski(400 , 5) from turtle import * def Sierpinski(length, level):     speed(0) # Fastest speed.     if level==0:         return     begin_fill() # Fill shape.     color("red")      for i in range(3):         Sierpinski(length/2,level-1)         fd(length)         lt(120) # Left turn 120 degrees.     end_fill()</pre> |  <p><b>Figure 3:</b> The Sierpiński triangle.</p> |
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**Problem 3:** Edit Program 3 to plot a Sierpiński square, where the middle square is removed at each stage. See one face of the Menger Sponge in Figure 4b.

**Example 4.** Use train tickets to construct stage 1 of the *Menger Sponge* (see Figures 4a and 4b).

**Problem 4:** Given that you need 6 tickets to make one small block:

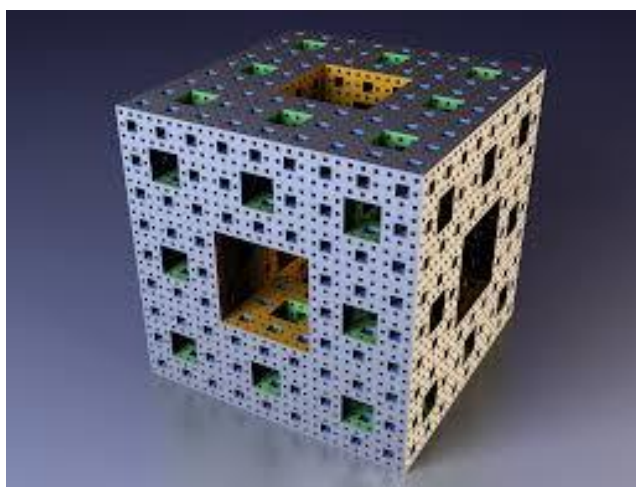
How many tickets do you need for the stage 1 construction? (See Figure 4a).

How many tickets would be required for the stage 2 construction?

How many tickets would be needed for the stage 4 construction? (See Figure 4b).

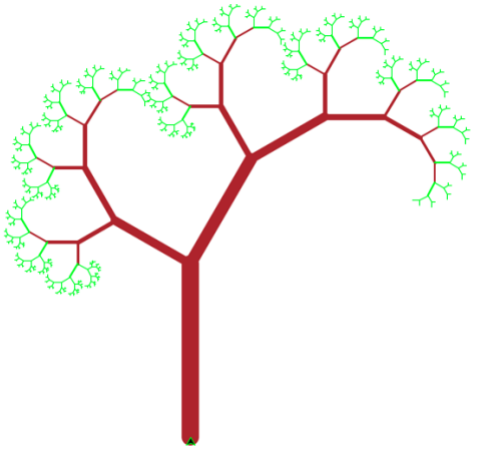


**Figure 4a:** Menger sponge, stage 1.



**Figure 4b:** Menger sponge, stage 4.

**Example 5.** A Python program for plotting a fractal tree.

|  |  |
|--|--|
| <pre>APPLE IDLE File Edit Format Run Options Window Help FractalTreeColor.py - /Users/99900361/Documents/2022 Files/Ste # Program 3: A program to plot a color fractal tree. # Run Module and type &gt;&gt;&gt; FractalTreeColor(200,10) in Python Shell. from turtle import * setheading(90)           # The turtle points straight up. penup()                 # Lift the pen. setpos(0,-250)          # Set initial point. pendown()               # Pen down.  def FractalTreeColor(length, level):     """     Draws a fractal tree     """     pensize(length/10)   # Thickness of lines.     if length &lt; 20:         pencolor("green")     else:         pencolor("brown")      speed(0)     if level &gt; 0:         fd(length)       # Forward.         rt(30)           # Right turn 30 degrees.         FractalTreeColor(length*0.7, level-1)         lt(90)           # Left turn 90 degrees.         FractalTreeColor(length*0.5, level-1)         rt(60)           # Right turn 60 degrees.         penup()         bk(length)      # Backward.         pendown()</pre> |  |
|--|--|

**Problem 5:** Edit Program 4 to plot a trifurcating tree with three branches at each level.

The formula for working out the fractal dimension  $D_f$ , say, is

$$D_f = - \frac{\ln(\text{Number of segments})}{\ln(\text{Length scale})}.$$

The fractal dimension of the Cantor set is  $D_f = \frac{\ln 2}{\ln 3} \sim 0.6309$ .

The fractal dimension of the Koch curve is  $D_f = \frac{\ln 4}{\ln 3} \sim 1.2619$ .

The fractal dimension of the Sierpiński triangle is  $D_f = \frac{\ln 3}{\ln 2} \sim 1.5850$ .

The fractal dimension of the Menger sponge is  $D_f = \frac{\ln 20}{\ln 3} \sim 2.7268$ .

**Problem 6:** Work out the fractal dimensions of the other fractals.

**Further Information:**

URL to download IDLE Python (which is free):

<https://www.python.org/downloads/>

URL for an introduction to the Python Turtle module:

<https://docs.python.org/3/library/turtle.html>

Python for A-Level Maths, undergraduate Maths and employability:

<https://www.mathscareers.org.uk/python-for-a-level-maths-undergraduate-maths-and-employability/>

Python for A-Level Maths and Beyond:

[https://drstephenlynch.github.io/webpages/Python\\_for\\_A\\_Level\\_Mathematics\\_and\\_Beyond.html](https://drstephenlynch.github.io/webpages/Python_for_A_Level_Mathematics_and_Beyond.html)

### The Mandelbrot Set

URL for the Mandelbrot Set (deep zoom):

<https://www.youtube.com/watch?v=LhOSM6uCWxk>

URL for the Mandelbrot Set song:

<https://www.youtube.com/watch?v=alj3OSoIDM>

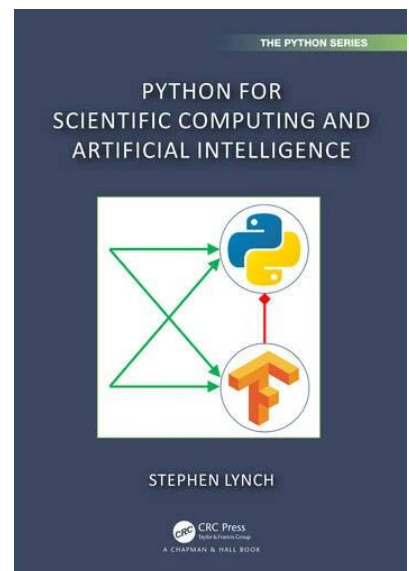
### IMA Workshops and My Python Books

One-day interactive workshops: Python for A-Level Mathematics and Beyond! <https://ima.org.uk/events/>

### NEW BOOK: Python for Scientific Computing and Artificial Intelligence

CRC Press

<https://www.routledge.com/Python-for-Scientific-Computing-and-Artificial-Intelligence/Lynch/p/book/9781032258713#>



Springer International Publishing:

<https://www.springer.com/us/book/9783319781440>

Jupyter Notebook:

[https://drstephenlynch.github.io/webpages/DSAP\\_Jupyter\\_Notebook.html](https://drstephenlynch.github.io/webpages/DSAP_Jupyter_Notebook.html)

GitHub:

<https://github.com/DrStephenLynch/Tekbac>

